THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Assignment 6
Due Date: 17 Apr 2018 (Tuesday)

1. Evaluate the following integrals.
   (a) $\int \sin^2 2x \sin 5x \, dx$
   (b) $\int \cos^2 2x \sin^3 2x \, dx$
   (c) $\int \frac{x-2}{\sqrt{x^2-4x+3}} \, dx$
   (d) $\int \frac{e^{x-1}}{1+e^{2x}} \, dx$
   (e) $\int x \sin^{-1} x \, dx$
   (f) $\int \cos(\ln x) \, dx$
2. Evaluate the following integrals.
   (a) $\int_4^6 |2x-1| \, dx$
   (b) $\int_0^{2\pi} |1+2\cos x| \, dx$
   (c) $\int_1^e x^2 \ln x \, dx$
3. By considering suitable integrals, evaluate the following limits.
   (a) $\lim_{n\to\infty} \frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n(n+1)}} + \cdots + \frac{1}{\sqrt{n(2n-1)}}$
   (b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$
4. Find $\frac{dy}{dx}$ if
   (a) $y = \int_x^{\sin x} \sin(e^t) \, dt$
   (b) $y = \int_0^x \sin(e^x + e^t) \, dt$
   (Hint: Using compound angle formula to expand $\sin(e^x + e^t)$.)
   (c) $y = \int_1^x \frac{e^{xt}}{t^2} \, dt$
   (Hint: Let $u = xt$.)
5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $a \in \mathbb{R}$. Show that
$$\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx.$$
   Hence, evaluate $\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x} \, dx$.

6. (a) Let $f, g : [0, a] \to \mathbb{R}$ be two continuous functions that satisfy

$$f(x) = f(a - x) \qquad \text{and} \qquad g(x) + g(a - x) = M,$$

where M is a real constant. Show that

$$\int_0^a f(x)g(x)\,dx = \frac{M}{2}\int_0^a f(x)\,dx.$$

(b) Hence, evaluate $\displaystyle\int_0^{\pi} x\cos^2 x \sin^4 x\,dx$.

7. Show that for all $x > 0$,

$$e^x - 1 \le \int_0^x \sqrt{e^{2t} + 1}\,dt \le \sqrt{2}(e^x - 1).$$

8. (a) Let $f(x)$ and $g(x)$ be two continuous functions on $[a, b]$. For $x \in [a, b]$, let

$$F(x) = \left(\int_a^x [f(t)]^2\,dt\right)\left(\int_a^x [g(t)]^2\,dt\right) - \left(\int_a^x f(t)g(t)\,dt\right)^2.$$

Show that $F(x)$ is increasing on $[a, b]$ and hence deduce that

$$\left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right) \ge \left(\int_a^b f(x)g(x)\,dx\right)^2.$$

(b) Using the result in (a), or otherwise, show that

$$\ln\left(\frac{p}{q}\right) \le \frac{p - q}{\sqrt{pq}},$$

where $0 < q \le p$.